JINR E2-82-159
February 26,1982
JINR Rapid Communications
4[78]-96,9 (1996)

OPTIMIZED LAMBDA-PARAMETRIZATION FOR THE QCD RUNNING COUPLING CONSTANT IN SPACELIKE AND TIMELIKE REGIONS*

A.V. Radyushkin**

** The investigation has been performed (and completed in February 1982) at the Laboratory of Theoretical Physics, JINR, Dubna, Russian Federation*

*** Present address: Physics Department, Old Dominion University, Norfolk, VA 23529, USA and Theory Group, Jefferson Lab, Newport News, VA 23606, USA*

The algorithm is described that enables one to perform an explicit summation of all the $(\pi^2/\ln(Q^2/\Lambda^2))^N$ corrections to $\alpha_s(Q^2)$ that appear owing to the analytic continuation from spacelike to timelike region of momentum transfer.

I. INTRODUCTION

Perturbative QCD is intensively applied now {1} to various processes involving large momentum transfers, both in spacelike ($q^2 = -Q^2 < 0$) and timelike ($q^2 > 0$) regions (for a review see [1]– [3]). However, the coupling constant $g(\mu)$ (i.e., the expansion parameter) is defined usually with the reference to some Euclidean (spacelike) configuration of momenta of scale μ. For spacelike q this produces no special complications. One simply uses the renormalization group to sum up the logarithmic corrections $(g^2(\mu)\ln(Q^2/\mu^2))^N$ that appear in higher orders and arrives at the expansion in the effective coupling constant $\alpha_s(Q^2)$ which in the lowest approximation is given by the famous asymptotic freedom formula [1]

$$\alpha_s(Q^2) = \frac{4\pi}{(11 - 2N_f/3)\ln(Q^2/\Lambda^2)} \ , \tag{1}$$

where Λ is the "fundamental" scale of QCD. In general, the Λ-parametrization of $\alpha_s(Q^2)$ is a series expansion in $1/L$ (where $L = \ln(Q^2/\Lambda^2)$), and the definition of Λ is fixed only if the $O(1/L^2)$-term is added to Eq.(1) [4].

For timelike q there appear, however, $i\pi$-factors ($\ln(Q^2/\mu^2) \to \ln(Q^2/\mu^2) \pm i\pi$), and it is not clear a priori what is the effective expansion parameter in this region. This problem was discussed recently {1} in a very suggestive paper by Pennington and Ross [5]. These authors analyzed the ratio $R(q^2) = \sigma(e^+e^- \to \text{hadrons})/\sigma(e^+e^- \to \mu^+\mu^-)$ for which the analytic continuation from the spacelike to timelike region is well-defined and investigated which of the three ansätze ($\alpha_s(q^2)$, $|\alpha_s(-q^2)|$ and $\mathrm{Re}\,\alpha_s(-q^2)$) better absorbs the $(\pi^2/L^2)^N$-corrections* in the timelike region $q^2 > 0$. Their conclusion was that $|\alpha_s(-q^2)|$ is better than $\alpha_s(q^2)$ and $\mathrm{Re}\,\alpha_s(-q^2)$. Nevertheless, it is easy to demonstrate by a straightforward calculation that $|\alpha_s(-q^2)|$ cannot absorb all the $(\pi^2/L^2)^N$-terms associated with the analytic continuation of the $\ln(Q^2/\mu^2)$-factors. Our main goal in the present letter is to show that by using the Λ-parametrization for $\alpha_s(Q^2)$ in the spacelike region it is possible to construct for $R(q^2)$ in the timelike region the expansion in which all the $(\pi^2/L^2)^N$-terms are summed explicitly.

II. Λ-PARAMETRIZATION IN SPACELIKE REGION

The starting point for the Λ-parametrization is the Gell-Mann-Low equation taken as a series expansion in $G = \alpha_s/4\pi$:

$$L \equiv \ln(Q^2/\Lambda^2) = \frac{1}{b_0 G} + \frac{b_1}{b_0^2}\ln G + \Delta + \frac{b_2 b_0 - b_1^2}{b_0^3} G + O(G^2) \ , \tag{2}$$

where b_k are β-function coefficients:

$b_0 = 11 - 2N_f/3$ [1], $b_1 = 102 - 38N_f/3$ [6], $b_2^{MS} = 2857/2 - 5033N_f/18 + 325N_f^2/54$ [7].

The parameter Δ in Eq.(2) is due to the lower boundary of the GML integral [8,9]. By a particular choice of Δ one fixes the definition of Λ: $\Lambda = \Lambda(\Delta)$ †. Eq.(2) is solved by iterations and the result is reexpanded in $1/L$:

$$\alpha_s(Q^2) = \frac{4\pi}{b_0 L}\left\{1 - \frac{L_1}{L} + \frac{1}{L^2}\left[L_1^2 - \frac{b_1}{b_0^2}L_1 + \frac{b_2 b_0 - b_1^2}{b_0^4}\right] + O(1/L^3)\right\} \ , \tag{3}$$

where

$$L_1 = \frac{b_1}{b_0^2}\ln(b_0 L) - \Delta \ . \tag{4}$$

The expansion (3) is useful, of course, only if it converges rapidly enough. In fact, the convergence of the $1/L$ series depends (i) on the value of L we are interested in and (ii) on the choice of Δ.

We emphasize that the most important for perturbative QCD is the region $L > 3$, since $L = 3$ corresponds to $\alpha_s \sim 0.5$, and the reliability of perturbation theory for larger α_s is questionable. Hence, in a realistic situation the

---

*Odd powers of $(i\pi/L)$ cancel because R is real.

†Of course, Λ depends also on the renormalization scheme chosen.

naive expansion parameter $1/L$ is smaller than (but usually close to) one third. Of course, $1/3$ is not very small, so one must check the coefficients of the $1/L$ expansion more carefully. First, there is a Δ-convention-independent term $(b_2 b_0 - b_1^2)/(b_0^4 L^2)$ which reduces for $N_f = 3$ to roughly $0.25/L^2$ and gives, therefore, less than 3%-correction to the simplest formula (1). There are also Δ-dependent terms like $L_1/L, L_1/L^2$ and one should choose Δ so as to minimize the upper value of the ratio L_1/L in the L-region of interest.

If one takes, e.g.,

$$\Delta = \Delta_{opt} = (b_1/b_0^2)\ln(4b_0)$$

then $L_1 = (b_1/b_0^2)\ln(L/4)$ and the ratio L_1/L is smaller than 7% in the whole region $L > 3$. Another choice [10] is to take

$$\Delta = \Delta(Q_0^2) = (b_1/b_0^2)\ln(b_0 L_0),$$

where $L_0 = \ln(Q_0^2/\Lambda^2)$ and Q_0^2 lies somewhere in the middle of the Q^2-region analyzed. In this case $L_1 = (b_1/b_0^2)\ln(L/L_0)$, i.e., L_1/L is zero for $Q^2 = Q_0^2$ and smaller than 7% for all Q^2 in the region where $L > 3$. An important observation is that both the choices minimize the corrections not only in Eq. (3) but also in the GML equation (2).

Really, for small G the only dangerous term in Eq. (2) is $\ln G$, hence, the best thing to do is to compensate it by taking $\Delta = -(b_1/b_0^2)\ln \overline{G}$, where $\overline{G}$ is $\alpha_s(Q^2)/4\pi$ averaged (in some sense) over the relevant Q^2-region. After this has been done, one may safely solve Eq. (2) by iterations and perform the $1/L$-expansion. For a proper choice of Δ Eq. (3) has 1% accuracy for $L > 3$, and, moreover, the total correction to the simplest formula (1) is less than 10%. However, accepting the most popular prescription

$$\Delta_{pop} = (b_1/b_0^2)\ln(b_0) = \Delta(Q_0^2 = e\Lambda^2)$$

(the only motivation for Δ_{pop} being the "aesthetic" criterion that L_1 should have the shortest form $L_1 = (b_1/b_0^2)\ln(L)$) one minimizes L_1/L in the region $Q^2 \sim 3\Lambda^2$ nobody is really interested in. Moreover, in the important region $L \sim 3$ one has $L_1^{pop}/L \sim 1/3$, and the convergence of the $1/L$ series is very poor in this case.

Thus, the Λ-parametrization (Eq.(3)) gives a rather compact and sufficiently precise expression for the effective coupling constant in the spacelike region provided a proper choice of the Δ-parameter has been made.

III. Λ-PARAMETRIZATION AND $\mathbf{R(e^+e^- \to hadrons, s)}$

The standard procedure (see, e.g., [11] and references therein) is to calculate the derivative $D(Q^2) = Q^2 dt/dQ^2$ of the vacuum polarization $t(Q^2)$ related to R by

$$R(s) = \frac{1}{2\pi i}\left(t(-s+i\epsilon) - t(-s-i\epsilon)\right) \ . \tag{5}$$

In perturbative QCD $D(Q^2)$ is given by the $\alpha_s(Q^2)$-expansion:

$$D(Q^2) = \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s(Q^2)}{\pi} + d_2\left(\frac{\alpha_s(Q^2)}{\pi}\right)^2 + d_3\left(\frac{\alpha_s(Q^2)}{\pi}\right)^3 + \ldots \right\} \ . \tag{6}$$

Only d_2 is known now [11,12] {2}, its value depending on the renormalization scheme chosen. Using Eq. (5) and the definition of D, one can relate $R(s)$ (or, more precisely, its perturbative QCD version $R^{QCD}(s)$) directly to $D(Q^2)$

$$R^{QCD}(s) = \frac{1}{2\pi i}\int_{-s-i\epsilon}^{-s+i\epsilon} D(\sigma)\frac{d\sigma}{\sigma} \ . \tag{7}$$

Integration in Eq.(7) goes below the real axis from $-s - i\epsilon$ to zero {3} and then above the real axis to $-s + i\epsilon$.

In a shorthand notation $D \Rightarrow R \equiv \Phi[D]$. In some important cases the integral can be calculated explicitly {4} :

$$1 \Rightarrow 1 \tag{8}$$

$$\frac{1}{L_\sigma} \Rightarrow \frac{1}{\pi}\arctan(\pi/L_s) = \frac{1}{L_s}\left\{1 - \frac{1}{3}\frac{\pi^2}{L_s^2} + \ldots\right\} \tag{9}$$

$$\frac{\ln(L_\sigma/L_0)}{L_\sigma^2} \Rightarrow \frac{\ln(\sqrt{L_s^2+\pi^2}/L_0)-(L_s/\pi)\arctan(\pi/L_s)+1}{L_s^2+\pi^2} = \frac{L_s/L_0}{L_s^2}\left\{1-\frac{\pi^2}{L_s^2}+\dots\right\}+\frac{5}{6}\frac{\pi^2}{L_s^4}+\dots \tag{10}$$

$$\frac{1}{L_\sigma^2} \Rightarrow \frac{1}{L_s^2+\pi^2} = \frac{1}{L_s^2}\left\{1-\frac{\pi^2}{L_s^2}+\dots\right\} \tag{11}$$

$$\frac{1}{L_\sigma^n} \Rightarrow (-1)^n\frac{1}{(n-1)!}\left(\frac{d}{dL_s}\right)^{n-2}\frac{1}{L_s^2+\pi^2} = \frac{1}{L_s^n}\left\{1-\frac{\pi^2}{L_s^2}\frac{n(n+1)}{6}+\dots\right\} \tag{12}$$

where $L_s=\ln(s/\Lambda^2), L_\sigma=\ln(\sigma/\Lambda^2)$ and L_0 is a constant depending on the Δ-choice.

Using the Λ-parametrization for $\alpha_s(\sigma)$ and incorporating Eqs.(8)-(12) (as well as their generalizations for $\ln^2 L/L^3, \ln L/L^3$ etc.) produces the expansion for $R^{QCD}(s)$

$$R^{QCD}(s) = \sum_q e_q^2\left\{1+\sum_{k=1} d_k\Phi[(\alpha_s/\pi)^k]\right\} \tag{13}$$

in which all the $(\pi^2/L^2)^N$-terms are summed up explicitly.

IV. QUEST FOR THE BEST EXPANSION PARAMETER

Note that the expansion (13) is not an expansion in powers of some particular parameter since the application of the Φ-operation normally violates nonlinear relations: $\Phi[1/L^2] \neq (\Phi[1/L])^2$, etc. A priori, there are no grounds to believe that a power expansion is better than any other (say, Fourier). In fact, the expansion (13) converges better than the generating expansion (6) for $D(\sigma)$ because, as it follows from Eqs. (9)-(12), $\Phi[\alpha_s^N]$ is always smaller than α_s^N. Moreover, $(\Phi[\alpha_s^{N+1}])^{1/(N+1)} < (\Phi[\alpha_s^N])^{1/N}$, i.e., the effective expansion parameter decreases in higher orders. Thus, if one succeeded in obtaining a good α_s^N expansion for $D(\sigma)$ (with all d_N being small numbers), then the resulting $\Phi[\alpha_s^N]$ expansion for $R^{QCD}(s)$ is even better, and the best thing to do is to leave it as it is.

However, if one insists that the result for $R^{QCD}(s)$ should have a form of a power expansion, then the best expansion parameter is evidently $\Phi[\alpha_s/\pi]$ because the largest nontrivial (i.e., $O(\alpha_s/\pi)$) term of the expansion is reproduced in the exact form and only higher terms are spoiled. The analogue of the simplest Λ-parametrization for $\alpha_s(Q^2)$ (Eq.(1)) is then

$$\tilde{\alpha}_s(q^2) = \frac{4}{b_0}\arctan\left(\frac{\pi}{\ln(q^2/\Lambda^2)}\right) . \tag{14}$$

Using Eqs. (8)-(13) it is easy to realize that $\alpha_s(q^2)$ is really a bad expansion parameter, because if one reexpands $\tilde{\alpha}_s(q^2)$ in $\alpha_s(q^2)$, there appear terms with large coefficients

$$\tilde{\alpha}_s(q^2) = \alpha_s(q^2)\left\{1-\frac{1}{3}\left(\frac{\pi b_0}{4}\right)^2\left(\frac{\alpha_s(q^2)}{\pi}\right)^2+\dots\right\} \approx \alpha_s\left\{1-17\left(\frac{\alpha_s}{\pi}\right)^2+\dots\right\} . \tag{15}$$

If one reexpands $\tilde{\alpha}_s(q^2)$ in Re $\alpha_s(-q^2)$ then the corresponding coefficient is even 2 times larger, whereas if $\tilde{\alpha}_s(q^2)$ is reexpanded in $|\alpha_s(-q^2)|$, the coefficient is 2 times smaller. This observation is in full agreement with the result of ref. [5] quoted in the Introduction.

V. CONCLUDING REMARKS

It should be noted that the change of the expansion parameter as given by Eq. (15) affects only the $(\alpha_s/\pi)^3$ coefficient of the R^{QCD}-expansion which has not been calculated yet {2}. So, within the present-day accuracy, all expansions for R^{QCD} have the same coefficients. It is worth emphasizing, nevertheless, that the π^2/L^2 terms produce for $\alpha_s \gtrsim 0.3$ more than 20% correction to α_s, i.e., they are more important (for an optimal choice of the Δ-parameter) than the 2-loop corrections in Eq. (3).

To conclude, we have described the construction of an optimized (i.e. rapidly convergent) Λ-parametrization for the effective QCD coupling constant in the spacelike region, and then we used it to obtain the fastest convergent expansion for the timelike quantity $R^{QCD}(s)$. The technique outlined in the present paper can be applied also to other R^{QCD}-like quantities. Such quantities do appear, e.g., in the QCD sum rule approach [13] in which the analysis of hadronic properties is based on the study of vacuum correlators of various currents. They appear also

in an alternative approach [14] based on the finite-energy sum rules [15]. It should be stressed that in the latter approach the R^{QCD}-like quantities enter into the basic integral relation, and the analysis is most conveniently performed if one has a simple analytic expression similar to that described above.

ACKNOWLEDGEMENTS

I am grateful to A.V. Efremov and D.V. Shirkov for their interest in this work and support. I thank D.I. Kazakov, O.V. Tarasov and A.A. Vladimirov for useful discussions.

---

[1] Politzer H.D. Physics Reports, 1974, 14C, p. 129.

[2] Mueller A.H. Physics Reports, 1981, 73, p.237.
Reya E. Physics Reports, 1981, 69, p.195.

[3] Efremov A.V., Radyushkin A.V. Rivista del Nuovo Cimento, 1980,3,No 2.

[4] Bace M. Physics Letters, B, 1978,78, p.132.

[5] Pennington M.R., Ross G.G. Physics Letters B, 1981,102, p. 167.

[6] Caswell W. Phys. Rev. Letters, 1974, 33, p.244.
Jones D.R.T. Nucl. Physics B, 1974, 75, p. 531.
Egoryan E. Sh., Tarasov O.V. Teor. Mat. Fizila, 1979, 41, p. 26.

[7] Tarasov O.V., Vladimirov A.A., Zharkov A. Yu. Physics Letters, B, 1980, 93, p.429.

[8] Bardeen W.A. et al. Phys. Rev. D, 1978, 18, p. 3998.

[9] Vladimirov A.A. Yad. Fizika, 1980, 31, p. 1083.

[10] Abbott L.F. Phys. Rev. Letters, 1980, 44, p. 1569.

[11] Chetyrkin K.G., Kataev A.L., Tkachov F.V. Nucl. Physics B, 1980, 174, p. 345.

[12] Dine M., Sapirstein J. Phys. Rev. Letters, 1979, 43, p. 668.
Celmaster W., Gonsalvez R. Phys. Rev. D, 1980, 21, p. 3112.

[13] Shifman M.A., Vainshtein A.I., Zakharov V.I. Nucl. Physics B, 1979, 147, pp. 385, 448.

[14] Krasnikov N.V., Pivovarov A.A. Preprint NBI-HE-81-38, Niels Bohr Institute, Kopenhagen (October 1981).

[15] Chetyrkin K.G., Krasnikov N.V., Tavkhelidze A.N. Physics Letters B, 1978, 76, p. 83.

NOTES ADDED

{1} This paper was submitted in 1982 to Physics Letters B, but not accepted because the referee was not convinced that it needs a rapid publication. I was recommended to write a longer version and submit it to a regular journal. Unwisely, I did not do that. Still, though the paper existed in the preprint form only, it was known to experts in multiloop calculations. In particular, Eq.(15) was incorporated into the 4-loop calculation of $R^{QCD}(s)$ [16,17]. Later, some of my results were used or rediscovered in several publications including very recent ones (see, e.g.,[18-22]). In 1996, the paper was reprinted in JINR Rapid Communications [23], but the subsequent experience convinced me that the only way to make the paper accessible to interested readers (and hopefully get credit for its results) is to submit it to an e-print archive. The original version is reproduced above without any

changes. I only added references to short notes presented below. They contain updating and clarifying remarks.

{2} The coefficient $d_3^{\overline{MS}}$ was calculated in refs. [16,17]. Starting from this level, the coefficients r_k of the $\alpha_s(s)$-expansion for $R^{QCD}(s)$

$$R^{QCD}(s) = \sum_q e_q^2 \left\{ 1 + \frac{\alpha_s(s)}{\pi} + r_2 \left(\frac{\alpha_s(s)}{\pi}\right)^2 + r_3 \left(\frac{\alpha_s(s)}{\pi}\right)^3 + \dots \right\}$$

differ from d_k: according to our Eq.(15) $r_3 = d_3 - (\pi b_0)^2/48$.

{3} This is an incorrect description of the actual straightforward procedure which I used to get results displayed in Eqs. (8)-(12). The central idea of the paper is to represent $D(\sigma)$ as a sum of terms for each of which the integral (7) can be calculated as an explicit analytic expression like $\ln \ln \sigma$ and then simply take the difference of these integrals at $-s + i\epsilon$ and $-s - i\epsilon$ using the standard prescription that the cut of $\ln z$ is on the negative real axis. This is precisely what one should do to analytically continue $t(Q^2)$ from the deep spacelike region. However, for terms in $D(\sigma)$ containing poles at $\sigma = \Lambda^2$ this prescription is equivalent to integration from $-s - i\epsilon$ below the real axis to some real point $\sigma_0 > \Lambda^2$ (rather than zero) and then above the real axis to $-s + i\epsilon$.

{4} In 1982, the applications of perturbative QCD in the $0 < s < \Lambda^2$ region were not treated as reliable, so $s > \Lambda^2$ (i.e. $L_s > 0$) is implied in Eqs. (8)-(12). Furthermore, the results are presented in a form most suitable for the $1/L_s$ expansion. To reconstruct the original expressions valid both for positive and negative L_s one should change

$$\arctan(\pi/L_s) \to \pi/2 - \arctan(L_s/\pi)$$

in Eqs. (8)-(12), but this form is not very illuminative for large L_s.

REFERENCES ADDED

[16] S.G. Gorishnii, A.L. Kataev, S.A. Larin, Phys.Lett. B259, 144 (1991).
[17] L.R. Surguladze, M. A. Samuel, Phys.Rev.Lett. 66, 560 (1991).
[18] A.A. Pivovarov, Nuovo Cim. 105A, 813 (1992).
[19] H.F. Jones, I.L. Solovtsov, Phys.Lett. B349, 519 (1995), hep-ph/9501344.
[20] K.A. Milton, O.P. Solovtsova, Phys.Rev. D57, 5402 (1998), hep-ph/9710316.
[21] B.V. Geshkenbein, B.L. Ioffe, hep-ph/9906406.
[22] D.V. Shirkov, I.L. Solovtsov, hep-ph/9906495.
[23] A.V. Radyushkin, JINR Rapid Comm. 4[78]-96,9 (1996).